

A Construction of Weakly and Non-Weakly Regular Bent Functions

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Abstract

In this article a technique for constructing p -ary bent functions from near-bent functions is presented. Two classes of quadratic p -ary functions are shown to be near-bent. Applying the construction of bent functions to these classes of near-bent functions yields classes of non-quadratic bent functions. We show that one construction in even dimension yields weakly regular bent functions. For other constructions, we obtain both weakly regular and non-weakly regular bent functions. In particular we present the first known infinite class of non-weakly regular bent functions.

Keywords: Bent function, Near-bent function, Semi-bent function, Weakly regular, Non-weakly regular, Fourier transform.

1 Introduction

Let p be a prime, and let V_n be any n -dimensional vector space over \mathbb{F}_p . For a function f from V_n to \mathbb{F}_p the *Fourier transform (or Walsh transform)* of f is the complex valued function \hat{f} on V_n given by

$$\hat{f}(b) = \sum_{x \in V_n} \epsilon_p^{f(x) - \langle b, x \rangle}$$

where $\epsilon_p = e^{2\pi i/p}$ and \langle, \rangle denotes any inner product on V_n . We say that f is a *bent* function if $|\hat{f}(b)|^2 = p^n$ for all $b \in V_n$. If $p = 2$ then $\epsilon_p = -1$ and $\hat{f}(b)$ is an integer, so a necessary condition for the existence of a bent function is that n is even. This does not hold for odd p , where bent functions can exist for both odd and even n . When p is odd, bent functions are sometimes called p -ary bent functions.

As all vector spaces of dimension n over \mathbb{F}_p are isomorphic we may associate V_n with the finite field \mathbb{F}_{p^n} . We then usually use the inner product $\langle x, y \rangle = \text{Tr}_n(xy)$ where $\text{Tr}_n(z)$ denotes the absolute trace of $z \in \mathbb{F}_{p^n}$. In

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this framework the Fourier transform of a function f from \mathbb{F}_{p^n} to \mathbb{F}_p is the complex valued function on \mathbb{F}_{p^n} given by

$$\widehat{f}(b) = \sum_{x \in \mathbb{F}_{p^n}} \epsilon_p^{f(x) - \text{Tr}_n(bx)}.$$

Often one considers the *normalized* Fourier coefficient $p^{-n/2} \widehat{f}(b)$ of a bent function. For any p , we can only say a priori that the normalized Fourier coefficients lie on the unit circle. For $p = 2$, a bent function must have normalized Fourier coefficients ± 1 , because the Fourier coefficients are real. For odd p , the values of the normalized Fourier coefficients of a bent function are also quite constrained, (cf. [2], [4, Property 8]). The possibilities are as follows:

$$p^{-n/2} \widehat{f}(b) = \begin{cases} \pm \epsilon_p^{f^*(b)} & : \quad n \text{ even or } n \text{ odd and } p \equiv 1 \pmod{4} \\ \pm i \epsilon_p^{f^*(b)} & : \quad n \text{ odd and } p \equiv 3 \pmod{4} \end{cases} \quad (1)$$

where f^* is a function from \mathbb{F}_{p^n} to \mathbb{F}_p that by definition gives the exponent of ϵ_p .

A bent function f is called *regular* if, for all $b \in \mathbb{F}_{p^n}$, we have

$$p^{-n/2} \widehat{f}(b) = \epsilon_p^{f^*(b)},$$

i.e., the coefficient of $\epsilon_p^{f^*(b)}$ is always $+1$. Thus the normalized Fourier coefficients of a regular bent function are a subset of (in fact the full set of) the p -th roots of unity. This is a natural generalization of the binary situation, where the normalized Fourier coefficients are ± 1 . It is obvious from equation (1) that regular bent functions can only exist for even n and for odd n with $p \equiv 1 \pmod{4}$. For example, when $p = 3$, regular bent functions can only exist in even dimensions, and the normalized Fourier coefficients are shown in the complex plane in Fig 1.

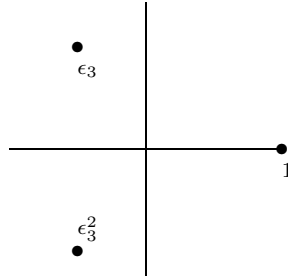


Fig 1, $p = 3$, even dimension

A bent function f is called *weakly regular* if, for all $b \in \mathbb{F}_{p^n}$, we have

$$p^{-n/2} \widehat{f}(b) = \zeta \epsilon_p^{f^*(b)}$$

for some complex number ζ with absolute value 1 (see [4]). By (1), ζ can only be ± 1 or $\pm i$. Thus the normalized Fourier coefficients of a weakly regular bent function are a rotation (through some multiple of $\pi/2$) of the p -th roots of unity. For $p = 3$, the three possibilities for the normalized Fourier coefficients are shown in Fig 2, Fig 3 and Fig 4. In this ternary case, weakly regular bent functions (that are not regular) can exist in both odd and even dimensions.

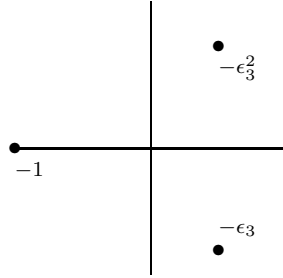


Fig 2, $p = 3$, even dimension

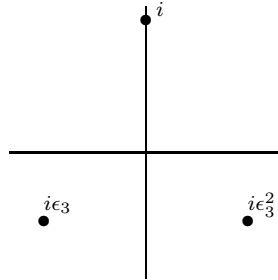


Fig 3, $p = 3$, odd dimension

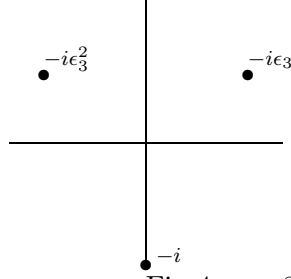


Fig 4, $p = 3$, odd dimension

Almost all known p -ary bent functions are weakly regular. Until this paper, there are just a few sporadic examples of non-weakly regular bent functions known (see [2, 3]). If $p = 3$, a non-weakly regular bent function in even dimensions would have normalized Fourier coefficients as in Fig 5 (Fig 1 and Fig 2 combined).

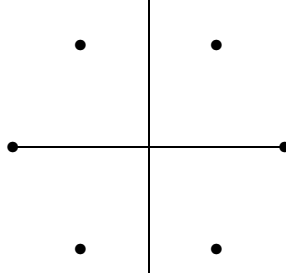


Fig 5, $p = 3$, even dimension

A ternary non-weakly regular bent function in odd dimensions would have normalized Fourier coefficients as in Fig 6 (Fig 3 and Fig 4 combined). We give a construction of such functions in this paper (for any p).

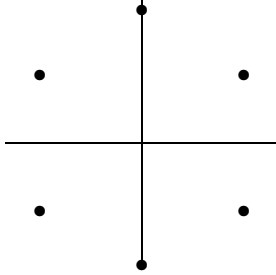


Fig 6, $p = 3$, odd dimension

For the binary case, where bent functions in odd dimension do not exist, the notion of *near-bent* functions was introduced in [5]. We generalize this now to characteristic p , and we call a function f from \mathbb{F}_{p^n} to \mathbb{F}_p *near-bent* if, for all $b \in \mathbb{F}_{p^n}$, $|\widehat{f}(b)|^2 = p^{n+1}$ or 0. We remark that in [1] the term semi-bent function is used for the same concept in characteristic 2.

In this article we first generalize to characteristic p the technique presented in [5] (see also [1]) that constructed binary bent functions from near-bent functions. In Section 2 we illustrate the principle of the construction. The idea is to choose near-bent functions in dimension n , which will be mappings from \mathbb{F}_{p^n} to \mathbb{F}_p , and glue them together using another copy of \mathbb{F}_p to obtain bent functions in dimension $n + 1$. These functions will be from $\mathbb{F}_{p^n} \times \mathbb{F}_p$ to \mathbb{F}_p . The near-bent functions must be chosen so that the supports of their Fourier transforms are disjoint, so exactly one of their Fourier transforms is nonzero at any point. We will show that varying the glueing coefficients from the extra copy of \mathbb{F}_p can drastically change the nature of the bent function, thereby demonstrating that inequivalent bent functions can be constructed by this method. For example, both weakly regular and non-weakly regular bent functions can be found by a simple tweak of the \mathbb{F}_p -coefficients (which are denoted c_k below, see Theorem 8).

In Section 3 we present a class of p -ary quadratic binomials that are near-bent. Another feature of our construction is that we obtain infinite families of *non*-quadratic bent functions from quadratic near-bent functions, so we are able to leave the quadratic world. The construction of bent functions from quadratic near-bent functions is described in detail in Section 4. In Section 5 we prove that one class of bent functions obtained with the near-bent functions introduced in Section 3 is weakly regular. For the general construction we give if and only if conditions to yield weakly regular bent

functions. Using these conditions we are then able to give infinite classes of non-weakly regular bent functions.

2 Obtaining bent functions from near-bent functions

Let f be a function from \mathbb{F}_{p^n} to \mathbb{F}_p , and \widehat{f} denote its Fourier transform. The support of \widehat{f} is then defined to be the set $\text{supp}(\widehat{f}) = \{b \in \mathbb{F}_{p^n} \mid \widehat{f}(b) \neq 0\}$. For any p -ary function f we have

$$\begin{aligned} \sum_{b \in \mathbb{F}_{p^n}} \left| \widehat{f}(b) \right|^2 &= \sum_{b \in \mathbb{F}_{p^n}} \sum_{x, y \in \mathbb{F}_{p^n}} \epsilon_p^{f(x) - \text{Tr}_n(bx) - (f(y) - \text{Tr}_n(by))} \\ &= \sum_{x, y \in \mathbb{F}_{p^n}} \epsilon_p^{f(x) - f(y)} \sum_{b \in \mathbb{F}_{p^n}} \epsilon_p^{\text{Tr}_n(b(y-x))}. \end{aligned}$$

Observing that $\sum_{b \in \mathbb{F}_{p^n}} \epsilon_p^{\text{Tr}_n(b(y-x))} = 0$ if $x \neq y$, and $\sum_{b \in \mathbb{F}_{p^n}} \epsilon_p^{\text{Tr}_n(b(y-x))} = p^n$ if $x = y$, we obtain the special case of *Parseval's relation*:

$$\sum_{b \in \mathbb{F}_{p^n}} \left| \widehat{f}(b) \right|^2 = \sum_{x, y \in \mathbb{F}_{p^n}, x=y} p^n = p^{2n}.$$

For a near-bent function f , clearly

$$\sum_{b \in \mathbb{F}_{p^n}} \left| \widehat{f}(b) \right|^2 = \left| \text{supp}(\widehat{f}) \right| p^{n+1}$$

and combining this with Parseval's relation gives

$$\left| \text{supp}(\widehat{f}) \right| = p^{n-1}.$$

The following theorem presents how to obtain p -ary bent functions from a set of p near-bent functions $f_0(x), f_1(x), \dots, f_{p-1}(x)$ from \mathbb{F}_{p^n} to \mathbb{F}_p with $\text{supp}(\widehat{f}_i) \cap \text{supp}(\widehat{f}_j) = \emptyset$ for $i \neq j$. We remark that then $\bigcup_{i=0}^{p-1} \text{supp}(\widehat{f}_i) = \mathbb{F}_{p^n}$.

Theorem 1 *Let $f_0(x), f_1(x), \dots, f_{p-1}(x)$ be near-bent functions from \mathbb{F}_{p^n} to \mathbb{F}_p such that $\text{supp}(\widehat{f}_i) \cap \text{supp}(\widehat{f}_j) = \emptyset$ for $0 \leq i \neq j \leq p-1$. Then the function $F(x, y)$ from $\mathbb{F}_{p^n} \times \mathbb{F}_p$ to \mathbb{F}_p defined by*

$$F(x, y) = (p-1) \sum_{k=0}^{p-1} \frac{y(y-1) \cdots (y-(p-1))}{y-k} f_k(x)$$

is bent.

Proof: For $(a, b), (x, y) \in \mathbb{F}_{p^n} \times \mathbb{F}_p$ the inner product we use is $\text{Tr}_n(ax) + by$. The Fourier transform \widehat{F} of F at (a, b) is

$$\begin{aligned}
\widehat{F}(a, b) &= \sum_{x \in \mathbb{F}_{p^n}, y \in \mathbb{F}_p} \epsilon_p^{F(x, y) - \text{Tr}_n(ax) - by} \\
&= \sum_{y \in \mathbb{F}_p} \epsilon_p^{-by} \sum_{x \in \mathbb{F}_{p^n}} \epsilon_p^{F(x, y) - \text{Tr}_n(ax)} \\
&= \sum_{y \in \mathbb{F}_p} \epsilon_p^{-by} \sum_{x \in \mathbb{F}_{p^n}} \epsilon_p^{(p-1)!(p-1)f_y(x) - \text{Tr}_n(ax)} \\
&= \sum_{y \in \mathbb{F}_p} \epsilon_p^{-by} \sum_{x \in \mathbb{F}_{p^n}} \epsilon_p^{f_y(x) - \text{Tr}_n(ax)} \\
&= \sum_{y \in \mathbb{F}_p} \epsilon_p^{-by} \widehat{f}_y(a).
\end{aligned}$$

As each $a \in \mathbb{F}_{p^n}$ belongs to the support of exactly one \widehat{f}_y , $y \in \mathbb{F}_p$, for this y we have $\left| \widehat{F}(a, b) \right| = |\epsilon_p^{-by} \widehat{f}_y(a)| = p^{\frac{n+1}{2}}$. \square

3 Monomial and binomial quadratic near-bent functions

Recall that a function f from \mathbb{F}_{p^n} to \mathbb{F}_p of the form

$$f(x) = \text{Tr}_n \left(\sum_{i=0}^l a_i x^{p^i+1} \right)$$

is called *quadratic*, its algebraic degree is two (if f is not constant), see [1, 2]. The following theorem giving the Fourier spectrum of a quadratic function in terms of the dimension of a certain subspace of \mathbb{F}_{p^n} (seen as a vector space over \mathbb{F}_p) is essentially Proposition 2 in [2]. The result is obtained via the standard squaring technique. We present the proof to keep the paper self contained.

Theorem 2 *Let f be the quadratic p -ary function*

$$f(x) = \text{Tr}_n \left(\sum_{i=0}^l a_i x^{p^i+1} \right),$$

and let $L(z)$ be the linearized polynomial

$$L(z) = \sum_{i=0}^l \left(a_i^{p^l} z^{p^{l+i}} + a_i^{p^{l-i}} z^{p^{l-i}} \right). \quad (2)$$

The square of the Fourier transform of f takes absolute values 0 and p^{n+s} , where s is the dimension of the kernel of the linear transformation on \mathbb{F}_{p^n} defined by $L(z)$.

Proof. With the standard squaring technique we obtain

$$\begin{aligned} |\widehat{f}(-b)|^2 &= \sum_{x, y \in \mathbb{F}_{p^n}} \epsilon_p^{f(x) - f(y) + \text{Tr}_n(b(x-y))} \\ &= \sum_{y, z \in \mathbb{F}_{p^n}} \epsilon_p^{f(y+z) - f(y) + \text{Tr}_n(bz)} \\ &= \sum_{z \in \mathbb{F}_{p^n}} \epsilon_p^{f(z) + \text{Tr}_n(bz)} \sum_{y \in \mathbb{F}_{p^n}} \epsilon_p^{f(y+z) - f(y) - f(z)}. \end{aligned}$$

Observe that

$$\begin{aligned} f(y+z) - f(y) - f(z) &= \text{Tr}_n \left(\sum_{i=0}^l a_i \left((y+z)^{p^{i+1}} - y^{p^{i+1}} - z^{p^{i+1}} \right) \right) \\ &= \text{Tr}_n \left(\sum_{i=0}^l a_i \left(yz^{p^i} + y^{p^i}z \right) \right) \\ &= \text{Tr}_n \left(y^{p^l} \sum_{i=0}^l \left(a_i^{p^l} z^{p^{l+i}} + a_i^{p^{l-i}} z^{p^{l-i}} \right) \right) \\ &= \text{Tr}_n(y^{p^l} L(z)). \end{aligned}$$

Consequently

$$\begin{aligned} |\widehat{f}(-b)|^2 &= \sum_{z \in \mathbb{F}_{p^n}} \epsilon_p^{f(z) + \text{Tr}_n(bz)} \sum_{y^{p^l} \in \mathbb{F}_{p^n}} \epsilon_p^{\text{Tr}_n(yL(z))} \\ &= p^n \sum_{\substack{z \in \mathbb{F}_{p^n} \\ L(z)=0}} \epsilon_p^{f(z) + \text{Tr}_n(bz)} \\ &= \begin{cases} p^{n+s} & \text{if } f(z) + \text{Tr}_n(bz) \equiv 0 \text{ on } \ker(L) \\ 0 & \text{otherwise} \end{cases} \quad (3) \end{aligned}$$

where in the last step we used that $f(z) + \text{Tr}_n(bz)$ is linear on the kernel of L . \square

3.1 Monomials

Our next goal is to find quadratic near-bent functions that can be used to construct bent functions as described in Theorem 1. We might hope for a monomial function, but unfortunately these do not exist as we now prove.

Theorem 3 *Quadratic monomial near-bent functions $f(x) = \text{Tr}_n(ax^{p^r+1})$, $a \in \mathbb{F}_{p^n}$, in odd characteristic p do not exist.*

Proof. The linearized polynomial (2) that corresponds to $f(x) = \text{Tr}_n(ax^{p^r+1})$ is given by $L(z) = az + a^{p^r}z^{p^{2r}}$. We have to show that for any odd prime p , integers $r, n \geq 1$ and $a \in \mathbb{F}_{p^n}$ the kernel of the linear map on \mathbb{F}_{p^n} induced by L does not have dimension 1. For a primitive element γ of \mathbb{F}_{p^n} let $a = \gamma^c$ for some $c, 0 \leq c \leq p^n - 2$. Then $L(\gamma^t) = 0$ for an exponent $t, 0 \leq t \leq p^n - 2$, if and only if

$$\gamma^{\frac{p^n-1}{2}-c(p^r-1)} = \gamma^{(p^{2r}-1)t},$$

which is equivalent to

$$\frac{p^n-1}{2} - c(p^r-1) \equiv (p^{2r}-1)t \pmod{p^n-1}.$$

Clearly the kernel of L has dimension 1 if and only if this congruence has $p-1$ incongruent solutions, which applies if and only if the two conditions $\gcd(p^{2r}-1, p^n-1) = p-1$ and $p-1$ divides $\frac{p^n-1}{2} - c(p^r-1)$ hold. The first condition is satisfied if and only if $\gcd(2r, n) = 1$, in particular n is then odd, which contradicts the second condition. \square

Remark 1 *In [2] it is pointed out that f is bent, i.e. the kernel of L has dimension 0, if and only if $p^{\gcd(2r,n)} - 1$ does not divide $\frac{p^n-1}{2} - c(p^r-1)$.*

3.2 Binomials

As a consequence of Theorem 3 we must consider non-monomial quadratic functions in order to be able to apply Theorem 1. Two classes of binomial near-bent functions are presented in the following theorem.

Theorem 4 Let $c \neq 0$ be an element of \mathbb{F}_p . The function f from \mathbb{F}_{p^n} to \mathbb{F}_p given by

(i)

$$f(x) = \text{Tr}_n \left(cx^{p^r+1} - cx^{p^t+1} \right) \quad (4)$$

is near-bent if and only if $\gcd(n, r+t) = \gcd(n, r-t) = \gcd(n, p) = 1$, and

(ii)

$$f(x) = \text{Tr}_n \left(cx^{p^r+1} + cx^{p^t+1} \right) \quad (5)$$

is near-bent if and only if $\gcd(n, 2(r+t)) = \gcd(n, 2(r-t)) = 2$, $r-t$ is odd, and $\gcd(n, p) = 1$.

Proof. By Theorem 2 a function f from \mathbb{F}_{p^n} to \mathbb{F}_p is near-bent if and only if the kernel (in \mathbb{F}_{p^n}) of the corresponding linearized polynomial $L(x)$ has dimension 1 as a vector space over \mathbb{F}_p , i.e., $\gcd(L(x), x^{p^n} - x)$ has degree p . Equivalently $\ker(L)$ is one-dimensional if and only if the associates $A(x)$ and $x^n - 1$ of $L(x)$ and $x^{p^n} - x$, respectively, satisfy $\deg(\gcd(A(x), x^n - 1)) = 1$, see [6, p.118].

For the binomial (4) we have $L(x) = c(x + x^{p^{2r}} - x^{p^{r-t}} - x^{p^{r+t}})$, consequently $A(x) = c(1 + x^{2r} - x^{r-t} - x^{r+t}) = c(x^{r+t} - 1)(x^{r-t} - 1)$. Using $\gcd(x^m - 1, x^n - 1) = x^{\gcd(m, n)} - 1$ we easily see that $\deg(\gcd(A(x), x^n - 1)) = 1$ if and only if $\gcd(n, r+t) = \gcd(n, r-t) = \gcd(n, p) = 1$. The last condition prevents 1 from being a multiple root of $x^n - 1$.

The polynomial $A(x)$ for the binomial (5) is given by $A(x) = c(1 + x^{2r} + x^{r-t} + x^{r+t}) = c(x^{r+t} + 1)(x^{r-t} + 1)$. Using $\gcd(x^m - 1, x^n - 1) = x^{\gcd(m, n)} - 1$ we obtain that

$$g = \gcd(x^{r \pm t} + 1, x^n - 1) = (x^{\gcd(2(r \pm t), n)} - 1) / (x^{\gcd(r \pm t, n)} - 1). \quad (6)$$

If n is odd then $g = 1$, thus we need n even and hence we have $\gcd(n, 2(r \pm t)) \geq 2$. If $\gcd(n, 2(r \pm t)) = 2$ then $r \pm t$ odd is a necessary and sufficient condition for $g = x + 1$. If $\gcd(n, 2(r \pm t)) = 2u$ for an odd integer $u > 1$ then by equation (6) we have $g = x^u + 1$, and $\gcd(n, 2(r \pm t)) = 2e$ for an even integer e yields $g = 1$ or $g = x^e + 1$. As a consequence, $\gcd(n, 2(r+t)) = \gcd(n, 2(r-t)) = 2$ and $r-t$ ($r+t$) odd are necessary conditions for $\gcd(A(x), x^n - 1) = x + 1$. As -1 is then a double root of $A(x)$ we obtain $\gcd(A(x), x^n - 1) = x + 1$ if and only if $\gcd(n, p) = 1$. \square

Remark 2 The kernel of $L(x)$ in \mathbb{F}_{p^n} for the function (4) is the set of the solutions of $x^p - x$, which is \mathbb{F}_p . For the function (5) the kernel of $L(x)$ in \mathbb{F}_{p^n} is the set of the solutions of $x^p + x$, which are all in \mathbb{F}_{p^2} .

Remark 3 Note that in part (i), n can be either even or odd, whereas in part (ii), the conditions imply that n must be even.

4 Constructions of Bent Functions, Examples

In order to apply Theorem 1 we need p near-bent functions such that the supports of their Fourier transforms are pairwise disjoint. We observe that the support of the Fourier transform of the quadratic p -ary function $f(x) = \text{Tr}_n(\sum_{i=0}^l a_i x^{p^i+1})$ is explicitly described in equation (3). This will be used in the following theorem which describes how to obtain a set of p quadratic near-bent functions with the required properties.

Theorem 5 Let g_0, g_1, \dots, g_{p-1} be quadratic near-bent functions from \mathbb{F}_{p^n} to \mathbb{F}_p such that the linearized polynomials L_0, L_1, \dots, L_{p-1} corresponding to g_0, g_1, \dots, g_{p-1} , respectively, have the same kernel $\{c\beta, 0 \leq c \leq p-1\}$ in \mathbb{F}_{p^n} . Let $b_0, b_1, \dots, b_{p-1} \in \mathbb{F}_{p^n}$ such that $g_k(\beta) + \text{Tr}_n(b_k\beta) = g_0(\beta) + k$, $0 \leq k \leq p-1$. The p near-bent functions f_0, f_1, \dots, f_{p-1} defined by $f_k(x) = g_k(x) + \text{Tr}_n(b_k x)$, $0 \leq k \leq p-1$, satisfy $\text{supp}(\widehat{f_i}) \cap \text{supp}(\widehat{f_j}) = \emptyset$ for $0 \leq i \neq j \leq p-1$.

Proof. We have to show that $-b \in \text{supp}(\widehat{f_j})$ implies $-b \notin \text{supp}(\widehat{f_i})$ for integers $0 \leq j, i \leq p-1$, $j \neq i$. Suppose $-b \in \text{supp}(\widehat{f_j})$, i.e. $g_j(\beta) + \text{Tr}_n(b_j\beta) + \text{Tr}_n(b\beta) = g_0(\beta) + j + \text{Tr}_n(b\beta) = 0$. Then $f_i(\beta) + \text{Tr}_n(b\beta) = g_i(\beta) + \text{Tr}_n(b_i\beta) + \text{Tr}_n(b\beta) = g_0(\beta) + i + \text{Tr}_n(b\beta) \neq 0$ when $i \neq j$. \square

Example 1. Let $g(x) = \text{Tr}_n(\sum_{i=0}^l a_i x^{p^i+1})$ be a quadratic near-bent function from \mathbb{F}_{p^n} to \mathbb{F}_p , let $\beta \in \mathbb{F}_{p^n}^*$ be a root of the corresponding linearized polynomial $L(x)$ and let $b_k \in \mathbb{F}_{p^n}$, $0 \leq k \leq p-1$, such that $\text{Tr}_n(b_k\beta) = k$. Then the function $F_1(x, y)$ from $\mathbb{F}_{p^n} \times \mathbb{F}_p$ to \mathbb{F}_p given by

$$F_1(x, y) = (p-1) \sum_{k=0}^{p-1} \frac{y(y-1) \cdots (y-(p-1))}{y-k} \left(\text{Tr}_n\left(\sum_{i=0}^l a_i x^{p^i+1}\right) + \text{Tr}_n(b_k x) \right)$$

is bent. Here all the $g_k(x)$ are equal to $g(x)$. As easily observed the bent function $F_1(x, y)$ is quadratic, a result of the fact that the p near-bent functions

$$f_k(x) = \text{Tr}_n\left(\sum_{i=0}^l a_i x^{p^i+1}\right) + \text{Tr}_n(b_k x)$$

used for the construction only differ in a linear term.

As the values of the Fourier spectrum of this quadratic bent function are the nonzero values in the Fourier spectrum of the underlying near-bent function, (1) implies that the nonzero normalized Fourier coefficients of a quadratic near-bent function f from \mathbb{F}_{p^n} to \mathbb{F}_p satisfy

$$p^{-n/2}\widehat{f}(b) = \begin{cases} \pm\epsilon_p^{J(b)} & : n \text{ odd and } p \equiv 3 \pmod{4} \\ \pm i\epsilon_p^{J(b)} & : n \text{ even or } n \text{ odd and } p \equiv 1 \pmod{4} \end{cases}$$

for some function $J(x)$ from $\text{supp}(\widehat{f})$ to \mathbb{F}_p . By [2, Proposition 1] quadratic bent functions are always (weakly) regular, and thus quadratic near-bent functions are also, in that sense that $\zeta p^{-n/2}\widehat{f}(b) = \epsilon_p^{J(b)}$ for all $b \in \text{supp}(\widehat{f})$ and a fixed complex number ζ with absolute value 1 (in this connection we remark that adding a linear term to a p -ary function f does not change the Fourier spectrum). A detailed description of the Fourier spectrum of quadratic near-bent functions will be given in Theorem 6.

As by Remark 2 the linearized polynomials of all near-bent functions of the form $x^{p^r+1} + x^{p^t+1}$ have the same kernel (the solutions of $x^p + x$) we now apply Theorem 5 to construct non-quadratic bent functions.

Example 2. Let $p = 3$ and $n = 8$. We construct a bent function in dimension 9. Let $g_0(x) = g_1(x) = \text{Tr}_n(x^{3^2+1} + x^{3+1})$, $g_2(x) = \text{Tr}_n(x^{3^6+1} + x^{3^5+1})$ be functions from \mathbb{F}_{3^8} to \mathbb{F}_3 . By the remark after Theorem 4 the kernel $\ker(L)$ of the corresponding linear transformations consists of the solutions of $x^3 + x$. For a root β of $x^2 + 1$ we have $g_0(\beta) = g_1(\beta) = g_2(\beta) = 0$ and therefore we need to find $b_0, b_1, b_2 \in \mathbb{F}_{3^8}$ such that $\text{Tr}_8(b_j\beta) = j$, to construct three near-bent functions such that the supports of their Fourier transforms are pairwise disjoint. Observing that $\text{Tr}_8(\beta) = 0$, $\text{Tr}_8(\beta^2) = 1$ and $\text{Tr}_8(2\beta^2) = 2$, we can choose $b_0 = 1$, $b_1 = \beta$, $b_2 = 2\beta$, and we therefore set $f_0(x) = \text{Tr}_8(x^{10} + x^4 + b_0x)$, $f_1(x) = \text{Tr}_8(x^{10} + x^4 + b_1x)$, $f_2(x) = \text{Tr}_8(x^{3^6+1} + x^{3^5+1} + b_2x)$. By Theorem 1, the following function from $\mathbb{F}_{3^8} \times \mathbb{F}_3$ to \mathbb{F}_3 of algebraic degree 4 is bent:

$$\begin{aligned} & 2 \sum_{k=0}^2 \frac{y(y-1)(y-2)}{y-k} f_k(x) \\ = & (2y^2 + 1)\text{Tr}_8(x^{10} + x^4 + x) + (2y^2 + 2y)\text{Tr}_8(x^{10} + x^4 + \beta x) + \\ & (2y^2 + y)\text{Tr}_8(x^{3^6+1} + x^{3^5+1} + 2\beta x) \\ = & 2y^2\text{Tr}_8(2x^{10} + 2x^4 + x^{3^6+1} + x^{3^5+1} + x) + \end{aligned}$$

$$\begin{aligned}
& y\text{Tr}_8(2x^{10} + 2x^4 + x^{3^6+1} + x^{3^5+1} + \beta x) + \text{Tr}_8(x^{10} + x^4 + x) \\
= & 2y^2\text{Tr}_8(2x^4 + x^{3^5+1} + x) + y\text{Tr}_8(2x^4 + x^{3^5+1} + \beta x) \\
& + \text{Tr}_8(x^{10} + x^4 + x).
\end{aligned}$$

Example 3. Let $g_0(x)$, $g_1(x)$, $g_2(x)$, b_0, b_1, b_2 be as in Example 2, except use $2g_1(x)$ in place of $g_1(x)$.

Example 4. Let $g_0(x)$, $g_1(x)$, $g_2(x)$, b_0, b_1, b_2 be as in Example 2, except take $g_1(x) = g_2(x)$ instead of taking $g_1(x)$ to be $g_0(x)$.

Example 5. Let $g_0(x)$, $g_1(x)$, $g_2(x)$, b_0, b_1, b_2 be as in Example 4, except use $2g_1(x)$ in place of $g_1(x)$.

Example 6. We give an example in even dimensions using Theorem 4 part (i); let $p = 3$ and $n = 5$. Let $g_0(x) = \text{Tr}_n(x^{3^2+1} - x^{3+1})$, $g_1(x) = \text{Tr}_n(2x^{3^2+1} - 2x^{3+1})$, $g_2(x) = \text{Tr}_n(x^{3^2+1} - x^{3+1})$ be functions from \mathbb{F}_{3^5} to \mathbb{F}_3 . Then these functions vanish on \mathbb{F}_3 , so we can choose $b_0 = 0, b_1 = 2, b_2 = 1$ which yields $f_0(x) = \text{Tr}_5(x^{10} - x^4)$, $f_1(x) = \text{Tr}_5(2x^{10} - 2x^4 + 2x)$, $f_2(x) = \text{Tr}_5(x^{10} - x^4 + x)$. The resulting bent function

$$2 \sum_{k=0}^2 \frac{y(y-1)(y-2)}{y-k} f_k(x) = 2(y-1)(y-2)f_0(x) + 2y(y-2)f_1(x) + 2y(y-2)f_2(x)$$

from $\mathbb{F}_{3^5} \times \mathbb{F}_3$ to \mathbb{F}_3 again has degree 4.

Remark 4 We will see later that Examples 2, 4, 6 are weakly regular, but examples 3 and 5 are not weakly regular.

5 (Non) Weak Regularity

We finally consider the question of whether the bent functions obtained from quadratic near-bent functions with Theorem 1, Theorem 4 and Theorem 5 are weakly regular. In this section we will prove necessary and sufficient conditions for weak regularity. Throughout this section, η denotes the quadratic character in \mathbb{F}_p .

5.1 Necessary and sufficient conditions for weak regularity, and the Fourier spectrum

We start with explicitly determining the Fourier spectrum of a quadratic near-bent function f . Choosing and fixing a basis $\{\alpha_1, \dots, \alpha_n\}$ of \mathbb{F}_{p^n} over

\mathbb{F}_p we correspond $x = \sum_{i=1}^n x_i \alpha_i$ to the vector $\mathbf{x} = (x_1, \dots, x_n)$. Then we can associate a quadratic function $f(x) = \text{Tr}_n(\sum_{i=0}^l a_i x^{p^i+1})$ with a quadratic form

$$f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

where \mathbf{x}^T denotes the transpose to the vector \mathbf{x} , and the matrix A has entries in \mathbb{F}_p . By [6, Theorem 6.21] any quadratic form is equivalent to a diagonal quadratic form, i.e. $D = C^T A C$ for a nonsingular matrix C over \mathbb{F}_p and a diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$. With standard arguments based on Theorems 6.26 and 6.27 in [6] one can express the Fourier transform of a quadratic near-bent function in terms of the product of the nonzero entries in D (for bent functions see [2, Proposition 1]).

Theorem 6 *Let f be a quadratic near-bent function from \mathbb{F}_{p^n} to \mathbb{F}_p and $f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ be the associated quadratic form. Then a corresponding diagonal matrix D has $n-1$ (not necessarily distinct) nonzero entries d_1, \dots, d_{n-1} , and the Fourier spectrum of f is given by*

$$\begin{aligned} \left\{ 0, \eta(\Delta) p^{\frac{n+1}{2}} \epsilon_p^{J(b)} \right\} & : p \equiv 1 \pmod{4}, \\ \left\{ 0, (-1)^{\frac{n-2}{2}} \eta(\Delta) i p^{\frac{n+1}{2}} \epsilon_p^{J(b)} \right\} & : p \equiv 3 \pmod{4} \text{ and } n \text{ even}, \\ \left\{ 0, (-1)^{\frac{n-1}{2}} \eta(\Delta) p^{\frac{n+1}{2}} \epsilon_p^{J(b)} \right\} & : p \equiv 3 \pmod{4} \text{ and } n \text{ odd}, \end{aligned}$$

where $J(x)$ is a function from $\text{supp}(\hat{f})$ to \mathbb{F}_p , $\Delta = \prod_{i=1}^{n-1} d_i$, and η denotes the quadratic character in \mathbb{F}_p .

Proof. We write $f(x) - \text{Tr}_n(bx) = j$ equivalently as

$$\mathbf{x}^T A \mathbf{x} - (\text{Tr}_n(b\alpha_1), \dots, \text{Tr}_n(b\alpha_n)) \mathbf{x} = \mathbf{x}^T A \mathbf{x} + c^T \mathbf{x} = j \quad (7)$$

where $c \in \mathbb{F}_p^n$. Denoting by $N_b(j)$, $j = 0, \dots, p-1$, the number of solutions in \mathbb{F}_p^n for (7) we observe that

$$\hat{f}(b) = \sum_{j=0}^{p-1} N_b(j) \epsilon_p^j.$$

Substituting $\mathbf{x} = C\mathbf{y}$ where C is the nonsingular matrix with $D = C^T A C$ we obtain for equation (7)

$$\mathbf{y}^T D \mathbf{y} + c^T C \mathbf{y} = \sum_{i=1}^n (d_i y_i^2 + c_i y_i) = j$$

with $c^T C = (c_1, \dots, c_n)^T \in \mathbb{F}_p^n$. Suppose that w.l.o.g. $d_n = 0$ is the only zero in the diagonal of D . Performing the substitution $y_i = z_i - c_i/(2d_i)$ for $i = 1, \dots, n-1$, we get

$$\sum_{i=1}^{n-1} d_i z_i^2 = j + \sum_{i=1}^{n-1} \frac{c_i^2}{4d_i} - c_n y_n. \quad (8)$$

We note that finding solutions $(z_1, \dots, z_{n-1}, y_n) \in \mathbb{F}_p^n$ for (8) is equivalent to finding solutions $(x_1, \dots, x_n) \in \mathbb{F}_p^n$ for (7), and the number of solutions is $N_b(j)$. Also note that the map $b \mapsto (c_1, \dots, c_n)$ is a bijection.

First suppose we have b with $c_n \neq 0$. For an arbitrary choice of j, z_1, \dots, z_{n-1} , equation (8) is satisfied for a unique choice for y_n . As a consequence $N_b(j)$ has the same value for each j and so $\widehat{f}(b) = 0$. (As an aside, there are $p^{n-1}(p-1)$ vectors (c_1, \dots, c_n) with $c_n \neq 0$, so $p^{n-1}(p-1)$ is the multiplicity of 0 in the Fourier spectrum.)

Now suppose we have b such that $c_n = 0$, and define $J(b) = -\sum_{i=1}^{n-1} \frac{c_i^2}{4d_i}$. We need to consider the cases of even and odd n separately. For even n , [6, Theorem 6.27] gives the number of solutions of (8) (i.e. gives $N_b(j)$) and we have

$$\begin{aligned} \widehat{f}(b) &= p \sum_{j=0}^{p-1} \left(p^{n-2} + p^{(n-2)/2} \eta \left((-1)^{(n-2)/2} (j - J(b)) \Delta \right) \right) \epsilon_p^j \\ &= p^{n/2} \eta \left((-1)^{(n-2)/2} \Delta \right) \sum_{j=0}^{p-1} \eta(j - J(b)) \epsilon_p^j. \end{aligned} \quad (9)$$

By [6, Theorem 5.15] we have

$$\sum_{j=0}^{p-1} \eta(j) \epsilon_p^j = \begin{cases} p^{1/2} & : p \equiv 1 \pmod{4}, \\ ip^{1/2} & : p \equiv 3 \pmod{4}, \end{cases}$$

(with the usual convention that $\eta(0) = 0$), thus equation (9) reduces to

$$\widehat{f}(b) = (-1)^{\frac{(p-1)(n-2)}{4}} \eta(\Delta) i^{s(p)} p^{(n+1)/2} \epsilon_p^{J(b)},$$

where $s(p) = 0$ if $p \equiv 1 \pmod{4}$ and else $s(p) = 1$.

For odd n , Theorem 6.26 in [6] implies that the values $N_b(j)$ are all equal except for $N_b(J(b))$ which differs from the others by $\eta((-1)^{(n-1)/2} \Delta) p^{(n+1)/2}$. Consequently

$$\widehat{f}(b) = (-1)^{\frac{(p-1)(n-1)}{4}} \eta(\Delta) p^{(n+1)/2} \epsilon_p^{J(b)},$$

which shows the correctness of the values for the Fourier transform given in the theorem.

Finally, as shown in [2], f is bent if and only if the associated quadratic form is nondegenerate. If the rank of A is $n-s$, and w.l.o.g. $d_1, \dots, d_{n-s} \neq 0$, we consider elements $b \in \mathbb{F}_{p^n}$ for which $c_{n-s+1} = \dots = c_n = 0$ (if one is nonzero, then $\widehat{f}(b) = 0$). Here we use that C is nonsingular and that $b \mapsto (\text{Tr}_n(b\alpha_1), \dots, \text{Tr}_n(b\alpha_n))$ defines a one-to-one linear transformation from \mathbb{F}_{p^n} to \mathbb{F}_p^n . Then with the same arguments as above we obtain that $|\widehat{f}(b)| = p^{(n+s)/2}$. Thus f is not near-bent for $s \neq 1$. \square

Remark 5 *We remark that the ‘aside’ comment in the previous proof gives a description of the b with $\widehat{f}(b) = 0$.*

As an immediate consequence we obtain the following corollary, which gives our necessary and sufficient conditions for weak regularity.

Corollary 1 *Let f_0, \dots, f_{p-1} be p -ary quadratic near-bent functions with $\text{supp}(\widehat{f}_i) \cap \text{supp}(\widehat{f}_j) = \emptyset$ for $0 \leq i \neq j \leq p-1$. Let A_i , $0 \leq i \leq p-1$, be the matrix of the quadratic form associated with f_i , and let Δ_i be the product of the nonzero eigenvalues of A_i , respectively. Then the bent function constructed as in Theorem 1 is weakly regular if and only if $\eta(\Delta_0) = \eta(\Delta_1) = \dots = \eta(\Delta_{p-1})$.*

5.2 (Non) Weak regularity of our examples

In order to decide whether the bent functions obtained in Theorem 5 using the functions of Theorem 4 are weakly regular, we are interested in the matrices (and their eigenvalues) of the quadratic forms associated with these quadratic functions. We start with a lemma on the quadratic character of the product of the nonzero eigenvalues of the matrix A associated with a quadratic near-bent function.

Lemma 1 *Let f be a quadratic near-bent function from \mathbb{F}_p^n to \mathbb{F}_p , let $\mathbf{x}^T A \mathbf{x}$ be the associated quadratic form and let Δ denote the product of the $n-1$ nonzero eigenvalues of A . For a nonzero constant $c \in \mathbb{F}_p$, the product $\Delta^{(c)}$ of the $n-1$ nonzero eigenvalues of the matrix for the quadratic form associated with cf satisfies*

$$\eta(\Delta^{(c)}) = \eta(c)^{n-1} \eta(\Delta).$$

Proof: The lemma follows with $cf(\mathbf{x}) = \mathbf{x}^T c A \mathbf{x}$. \square

We observe that as a consequence of Theorem 6 and Lemma 1, if n is even then we can change the signs of the Fourier coefficients if we switch from the near-bent function f to the near-bent function cf , with a nonsquare $c \in \mathbb{F}_p$. With these observations we can obtain an infinite class of non-weakly regular bent functions. As building blocks we may use the near-bent functions from Theorem 5.

Theorem 7 *Let n be even. For each $0 \leq k \leq p-1$ let c_k be an element of \mathbb{F}_p^* and let g_k be the near-bent function*

$$g_k(x) = c_k \text{Tr}_n(x^{p^r+1} - x^{p^t+1}) \quad \text{or} \quad g_k(x) = c_k \text{Tr}_n(x^{p^r+1} + x^{p^t+1})$$

from \mathbb{F}_{p^n} to \mathbb{F}_p described as in Theorem 4(i) or Theorem 4(ii), respectively. Here $r = r_k$ and $t = t_k$ may vary with k . Let f_0, \dots, f_{p-1} be p -ary quadratic near-bent functions from \mathbb{F}_{p^n} to \mathbb{F}_p with $\text{supp}(\widehat{f_i}) \cap \text{supp}(\widehat{f_j}) = \emptyset$ for $i \neq j$, which are obtained as in Theorem 5 from the binomial near-bent functions g_k . Then the bent function $F(x, y)$ from $\mathbb{F}_{p^n} \times \mathbb{F}_p$ to \mathbb{F}_p

$$F(x, y) = (p-1) \sum_{k=0}^{p-1} \frac{y(y-1) \cdots (y-(p-1))}{y-k} f_k(x)$$

is weakly regular for $(p-1)^p/2^{p-1}$ choices for $(c_0, \dots, c_{p-1}) \in (\mathbb{F}_p^)^p$, and non-weakly regular for the remaining $(2^{p-1}-1)(p-1)^p/2^{p-1}$ choices for $(c_0, \dots, c_{p-1}) \in (\mathbb{F}_p^*)^p$.*

Proof: Let Δ_k denote the product of the nonzero eigenvalues of the matrix that corresponds to g_k . Then by Corollary 1 and Lemma 1, $F(x, y)$ is weakly regular if and only if $\eta(c_0)^{n-1} \eta(\Delta_0) = \eta(c_1)^{n-1} \eta(\Delta_1) = \cdots = \eta(c_{p-1})^{n-1} \eta(\Delta_{p-1})$. Since $n-1$ is odd, for a fixed value of $\eta(c_k)^{n-1} \eta(\Delta_k)$, 1 or -1 , for every $0 \leq k \leq p-1$ we have $(p-1)/2$ choices for c_k . This gives in total $2((p-1)/2)^p$ choices for $(c_0, \dots, c_{p-1}) \in (\mathbb{F}_p^*)^p$ for which $F(x, y)$ is weakly regular. \square

The conditions in the previous theorem mean that a simple tweak of the coefficients c_k can drastically change the nature of the bent function. For example, we observe that the bent functions from Example 2 and 4 where each c_k is equal to 1, are weakly regular. The normalized Fourier spectrum

(without multiplicities) of Example 2 (and 4) is shown in Fig 4, and the spectrum with multiplicities is

$$(-i)^{2187}, (-i\epsilon_3)^{2268}, (-i\epsilon_3^2)^{2106},$$

(recall the dimension is 9). In Examples 3 and 5, still $c_0 = c_2 = 1$, but we changed c_1 to $c_1 = 2$. This alters the sign of the Fourier coefficients for one of the near-bent functions. Consequently the bent functions in Examples 3 and 5 are non-weakly regular. The normalized Fourier spectrum (without multiplicities) of Example 3 (and 5) is shown in Fig 6, and the spectrum with multiplicities is

$$(i\epsilon_3)^{702}, (i\epsilon_3^2)^{756}, (-i)^{1458}, i^{729}, (-i\epsilon_3^2)^{1404}, (-i\epsilon_3)^{1512}.$$

Many non-weakly regular bent functions can clearly be constructed in this manner; one simply has to arrange that for two of the coefficients c_k and $c_{k'}$ we have $\eta(c_k)\eta(\Delta_k) \neq \eta(c_{k'})\eta(\Delta_{k'})$.

Finally we remark that the arguments in Theorem 7 are not restricted to near-bent functions of the form (4), (5), but applicable to every set of p quadratic near-bent functions in even dimension satisfying the conditions of Theorem 1.

5.3 A Family of Weakly Regular Bent Functions

We show that the bent functions obtained with Theorem 1 using the near-bent functions (4) are always weakly regular when n is odd. As weakly regular bent functions are useful for the construction of certain combinatorial objects such as partial difference sets, strongly regular graphs and association schemes (see [7]) they are of independent interest.

First observe that a quadratic function $f(x) = \text{Tr}_n(\sum_{i=0}^l a_i x^{p^i+1})$ can be written in the form $f(x) = \text{Tr}_n(x\mathcal{L}(x))$ for the linear transformation $\mathcal{L}(x) = \sum_{i=0}^l a_i x^{p^i}$ on \mathbb{F}_{p^n} . As we did in Section 5.1, if we choose a basis $\{\alpha_0, \dots, \alpha_{n-1}\}$ of \mathbb{F}_{p^n} over \mathbb{F}_p , and let $x = \sum_{i=0}^{n-1} x_i \alpha_i \in \mathbb{F}_{p^n}$ correspond to the vector $\mathbf{x} = (x_0, \dots, x_{n-1})$, then we can associate $f(x)$ with a quadratic form $\mathbf{x}^T A \mathbf{x}$.

We now choose $\{\alpha_0, \dots, \alpha_{n-1}\}$ to be a self-dual basis of \mathbb{F}_{p^n} over \mathbb{F}_p , which exists if and only if n is odd. Then a straightforward calculation shows that A is the matrix representation of the linear transformation induced by $\mathcal{L}(x)$ with respect to the given self-dual basis.

For the functions of interest to us, $\mathcal{L}(x)$ is of the form

$$\mathcal{L}(x) = cx^{p^r} - cx^{p^t}$$

where $c \in \mathbb{F}_p$. We choose the self-dual basis to be also a normal basis $\{\alpha, \alpha^p, \dots, \alpha^{p^{n-1}}\}$ (which always is possible when n is odd, see [6]). Then $\mathcal{L}(\alpha_i) = \mathcal{L}(\alpha^{p^i}) = c\alpha^{p^{r+i}} - c\alpha^{p^{t+i}} = c\alpha_{r+i \bmod n} - c\alpha_{t+i \bmod n}$. Hence the corresponding matrix $A_c^{(r,t)}$ is an $n \times n$ circulant matrix with first column $(a_0, \dots, a_{n-1})^T$ with $a_r = c, a_t = -c$ and $a_i = 0, i \neq r, t$, or equivalently with first row (s_0, \dots, s_{n-1}) with $s_i = c$ if $i \equiv n - r \bmod n$, $s_i = -c$ if $i \equiv n - t \bmod n$, and $s_i = 0$ otherwise. Using the notation $A = C(s_0, \dots, s_{n-1})$ for a circulant matrix A with first row (s_0, \dots, s_{n-1}) we can summarize these observations with

$$\begin{aligned} A_c^{(r,t)} &= C(s_0, \dots, s_{n-1}) \quad \text{where} \\ s_i &= \begin{cases} c & : i \equiv n - r \bmod n \\ -c & : i \equiv n - t \bmod n \\ 0 & : \text{otherwise.} \end{cases} \end{aligned} \quad (10)$$

We will use the following result on eigenvalues of circulant matrices, see 1.6 in [8].

Lemma 2 *Let n be an integer relatively prime to p , u a primitive n -th root of unity over \mathbb{F}_p , and let $A = C(s_0, \dots, s_{n-1})$ be an $n \times n$ circulant matrix. The eigenvalues of A are given by*

$$\lambda_j = \sum_{i=0}^{n-1} s_i u^{ij}, \quad j = 0, \dots, n-1.$$

We denote the eigenvalues of the matrix $A_1^{(r,t)}$ corresponding to $x^{p^r} - x^{p^t} \in \mathbb{F}_{p^n}[x]$ as $\lambda_j^{(r,t)}$, $j = 0, 1, \dots, n-1$. By (10) and Lemma 2 we then have $\lambda_j^{(r,t)} = u^{(n-r)j} - u^{(n-t)j}$ for $j = 0, 1, \dots, n-1$, where u is a primitive n -th root of unity over \mathbb{F}_p . As easily seen $\lambda_0^{(r,t)} = 0$ and hence the product of the nonzero eigenvalues of $A_1^{(r,t)}$ is $\Delta^{(r,t)} = \prod_{j=1}^{n-1} \lambda_j^{(r,t)}$. Next we show that $\Delta^{(r,t)}$ does not depend on the special choice of r and t .

Lemma 3 *For an odd integer n let r, t and v, w be pairs of integers satisfying the conditions in Theorem 4(i). Then with the above notations*

$$\Delta^{(r,t)} = \prod_{j=1}^{n-1} \lambda_j^{(r,t)} = \prod_{j=1}^{n-1} \lambda_j^{(v,w)} = \Delta^{(v,w)}.$$

Proof: For fixed j , $0 \leq j \leq n-1$, we are interested in two integers $0 \leq k_j, c_j \leq n-1$ such that

$$\lambda_j^{(r,t)} = u^{(n-r)j} - u^{(n-t)j} = (u^{(n-v)k_j} - u^{(n-w)k_j})u^{c_j} = \lambda_{k_j}^{(v,w)}u^{c_j}.$$

We therefore consider the linear system in the two variables k_j, c_j

$$\begin{aligned} -rj &\equiv -vk_j + c_j \pmod{n} \\ -tj &\equiv -wk_j + c_j \pmod{n} \end{aligned}$$

yielding

$$k_j \equiv \frac{r-t}{v-w}j \pmod{n} \quad \text{and} \quad c_j \equiv -\frac{rw-tv}{v-w}j \pmod{n}.$$

We remark that k_j and c_j are well defined since $\gcd(v-w, n) = 1$. Moreover also $\gcd(r-t, n) = 1$ thus $(r-t)/(v-w)$ is an invertible residue modulo n and k_j runs through the integers modulo n if j does. Consequently

$$\begin{aligned} \Delta^{(r,t)} &= \prod_{j=1}^{n-1} \lambda_j^{(r,t)} = \prod_{j=1}^{n-1} (u^{(n-v)k_j} - u^{(n-w)k_j})u^{c_j} \\ &= \prod_{j=1}^{n-1} \lambda_{k_j}^{(v,w)} \prod_{j=1}^{n-1} u^{c_j} = \Delta^{(v,w)} \prod_{j=1}^{n-1} u^{c_j}. \end{aligned}$$

Since

$$\prod_{j=1}^{n-1} u^{c_j} = u^{\sigma \sum_{j=1}^{n-1} j} = 1,$$

where $\sigma \equiv -\frac{rw-tv}{v-w} \pmod{n}$, the proof is complete. \square

Theorem 8 *Let n be odd. For each $0 \leq k \leq p-1$ let c_k be an element of \mathbb{F}_p^* and let g_k be the near-bent function*

$$g_k(x) = c_k \text{Tr}_n(x^{p^r+1} - x^{p^t+1})$$

from \mathbb{F}_{p^n} to \mathbb{F}_p described as in Theorem 4(i). Here $r = r_k$ and $t = t_k$ may vary with k . Let f_0, \dots, f_{p-1} be p -ary quadratic near-bent functions from \mathbb{F}_{p^n} to \mathbb{F}_p with $\text{supp}(\widehat{f_i}) \cap \text{supp}(\widehat{f_j}) = \emptyset$ for $i \neq j$, which are obtained as in

Theorem 5 from the binomial near-bent functions g_k . Then the bent function $F(x, y)$ from $\mathbb{F}_{p^n} \times \mathbb{F}_p$ to \mathbb{F}_p

$$F(x, y) = (p-1) \sum_{k=0}^{p-1} \frac{y(y-1) \cdots (y-(p-1))}{y-k} f_k(x)$$

is weakly regular.

Proof. By Lemma 1 and Lemma 3 we may write $c_k^{n-1} \Delta$ for the product of the nonzero eigenvalues of the circulant matrix we correspond to g_k , $0 \leq k \leq p-1$. Since $n-1$ is even we have $\eta(c_k^{n-1} \Delta) = \eta(\Delta)$ for all $0 \leq k \leq p-1$. By Corollary 1 we obtain then the assertion of the theorem. \square

Remark 6 *Theorem 8 implies that Example 6 is a weakly regular bent function.*

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